JACOBI OSCULATING RANK AND ISOTROPIC GEODESICS ON NATURALLY REDUCTIVE 3-MANIFOLDS

J. C. GONZÁLEZ-DÁVILA

Abstract. We study the Jacobi osculating rank of geodesics on naturally reductive homogeneous manifolds and we apply this theory to the 3-dimensional case. Here, each non-symmetric, simply connected naturally reductive 3-manifold can be given as a principal bundle $M^3(\kappa,\tau)$ over a surface of constant curvature κ, such that the curvature of its horizontal distribution is a constant $\tau > 0$, with $\tau^2 \neq \kappa$. Then, we prove that the Jacobi osculating rank of every geodesic of $M^3(\kappa,\tau)$ is two except for the Hopf fibers, where it is zero. Moreover, we determine all isotropic geodesics and the isotropic tangent conjugate locus.

Keywords and phrases: Jacobi osculating rank, isotropic geodesic, isotropic conjugate point, homogeneous structure

2000 MSC: 53C20, 53C30, 53C22

Contents

1. Introduction

A Jacobi field V on a homogeneous Riemannian manifold (M,g) which is the restriction of a Killing vector field along a geodesic is called *isotropic* [21]. From the homogeneity of (M,g), it means that V is the restriction of a fundamental vector field of some element X in the Lie algebra of the isometry group $I(M,g)$ of (M,g). If moreover V vanishes at a point of the geodesic, then X belongs to the Lie algebra of the isotropy subgroup at this point. This particular situation was what originally motivated the term *isotropic* (see [3] and [4]).

Two points $p, q \in M$ are said to be *isotropically conjugate* if there exists a nonzero isotropic Jacobi field V along a geodesic passing through p and q such that V vanishes at these points. Clearly, they are isotropically conjugate along any geodesic joining p to q. When every Jacobi field vanishing at p and q is isotropic, we say that they are *strictly isotropic conjugate points.*

On symmetric spaces, the Jacobi equation has simple solutions and one directly obtains that *any pair of conjugate points in a Riemannian symmetric space are strictly isotropic* (see [12]). On [8, Remark 4.7] are constructed examples of 3-symmetric spaces admitting pairs of conjugate points which are isotropic but not strictly isotropic. A geodesic starting at a point $p \in M$ is said to be *isotropic* (resp., *strictly isotropic*) if each one of its conjugate points to p is isotropic (resp., strictly isotropic). Then, all geodesic on a symmetric space is strictly isotropic. In the case of a naturally reductive space, W. Ziller in [21] proposed to examine conjectures like: *A naturally reductive space with the property that all its geodesics are strictly isotropic is locally symmetric.* A positive answer to the conjecture for $n \leq 5$ is given in [12] and in [8], for naturally reductive compact 3-symmetric spaces.

The parallel translation of the Jacobi operator $R_{\gamma_u} := R(\gamma_u', \cdot)\gamma_u'$ along a geodesic γ_u starting at the origin o of a naturally reductive homogeneous manifold $(M = G/K, g)$ with $\gamma_u'(0) = u$, for some unit vector $u \in \mathfrak{m} \cong T_oM$, determines a curve $R_u(t)$ in the space of the self-adjoint operators $\mathcal{S}(\mathfrak{m})$ of $\mathfrak{m}$, which is an orbit of a one-parameter subgroup of isometries of this space. Then, as it is shown in Lemma 3.3, such a curve has constant osculating rank. We refer to such constant as the *Jacobi osculating rank of the geodesic* γ_u and it will be denoted by $\mathrm{rank}_{\mathrm{osc}}(u)$. Then, there exist smooth functions $a_1, \ldots, a_r$, where $r = \mathrm{rank}_{\mathrm{osc}}(u)$, such that

$$R_u(t) = R_u(0) + a_1(t)R_u'(0) + \cdots + a_r(t)R_u^{r)}(0),$$

$R_u'(0), \ldots, R_u^{r)}(0)$ are linearly independent in $\mathcal{S}(\mathfrak{m})$ and $r \leq \frac{n(n-1)}{2}$, $n = \dim M$. Moreover, in Lemma 3.7 one obtains that the Jacobi osculating rank is also the osculating rank of the curve obtained in $\mathcal{S}(\mathfrak{m})$ by parallel translation of the Jacobi operator for the *canonical connection* adapted to a reductive decomposition of $(M = G/K, g)$.

When the Jacobi osculating rank does not depend on the choice of the geodesic, the naturally reductive homogeneous space is said to have *constant Jacobi osculating rank*. Clearly, symmetric spaces have Jacobi constant osculating rank zero, or equivalently all these curves are constant, and for the simply connected case, in Theorem 3.6 the converse is stated. Then the notion of Jacobi osculating rank can be view as a natural way to 'measure' what a naturally reductive homogeneous space, or more general a g.o. space (see Remark 3.4) moves away from to be locally symmetric.

Some examples of non-symmetric naturally reductive spaces with constant Jacobi osculating rank are already known. Concretely, A. M. Naveira and A. Tarrío in [16] and, together with E. Macías, in [15] have proved that the Berger manifold $V_1 = Sp(2)/SU(2)$ and the Wilking manifold $V_3 = (SO(3) \times SU(3))/U^\bullet(2)$, endowed this last one with a particular bi-invariant metric, have constant Jacobi osculating rank two and, in the context of g.o. spaces, T. Arias-Marco and A. M. Naveira in [1] have shown that the Jacobi osculating rank of the six-dimensional Kaplan's example is constant equals four. (See [11], for a brief survey on isotropic Jacobi fields and Jacobi osculating rank and for further references.) For a general naturally reductive homogeneous space, the constancy of the Jacobi osculating rank is not necessarily satisfied. In fact, as a consequence from Proposition 3.9, one obtains that non-locally symmetric naturally reductive spaces of dimension $n \leq 5$, generalized Heisenberg

groups, Berger spheres or φ-symmetric spaces are some examples of non-constant Jacobi osculating rank.

In this article, we shall focus our attention on naturally reductive homogeneous spaces of dimension 3. The classification for the simply connected case is well known [20] (see also [6], [14]). They are the symmetric spaces $\mathbb{R}^3$, $S^3(c)$, $H^3(-c)$, $S^2(c)\times\mathbb{R}$ and $H^2(-c)\times\mathbb{R}$, where $c>0$, and unimodular Lie groups equipped with a left invariant metric such that the dimension of their isometry groups is four. Each one of these spaces fibers as an one-dimensional principal fiber bundle over a complete simply connected surface of constant curvature κ and the horizontal distribution of this fibration is the kernel of a connection form with constant curvature τ, which can be taken positive (see [9], [19]). The fibers are geodesics and there exists a one-parameter family of translations along the fibers, generated by a unit Killing vector field ξ, the *Hopf vector field*. Then these manifolds, which will be denoted by $M^3(\kappa,\tau)$, are classified, up to isometry, in terms of the pair (κ,τ), where $\kappa\neq\tau^2$ and $\tau>0$, into three types: Berger spheres $S^3(\kappa,\tau)$, if $\kappa>0$; the universal covering $\widetilde{SL_2}(\kappa,\tau)$ of $SL(2,\mathbb{R})(\kappa,\tau)$, if $\kappa<0$; and the Heisenberg group $H_3(\tau)$, if $\kappa=0$.

Because ξ is a unit Killing vector field, every geodesic γ_u on $M^3(\kappa,\tau)$ intersects each fiber with a constant slope angle $\theta=\mathrm{ang}(\xi_o,u)\in]0,\pi[$. Moreover, according with Lemma 4.2, every pair of geodesics starting at the origin with same slope angle are related under the isotropy action. It allows us to prove in Theorem 4.3 that *the Jacobi osculating rank of every geodesic on $M^3(\kappa,\tau)$ is two except for the Hopf fibers, where it is zero.* Moreover, for each slope angle $\theta\in]0,\pi[$, the curve $R_u(t)$ in $\mathbb{S}(\mathfrak{m})$ is a circle whose radius contracts to the point R_{ξ_o} as θ converges to 0 or to π.

S. Engel determined in [7] the conjugate radius and the cut locus of $M^3(\kappa,\tau)$, as generalisation of the results on Berger spheres carried out by Sasaki [18] and Rakotoniaina [17]. Here, we go forward with this research. First, taking into account that, by using the adapted canonical connection, the Jacobi equation can be expressed as a differential equation with constant coefficients, we obtain in Theorem 5.2 all isotropic conjugate points of $M^3(\kappa,\tau)$. Then its isotropic geodesics can be explicitly determined. They are one-to-one geodesics without pairs of conjugate points and with slope angle θ equals to $\pi/2$ for $H_3(\tau)$ and $\theta\in[\varepsilon,\pi-\varepsilon]$, where $\varepsilon=\arctan\tau/\sqrt{-k}$, for $\widetilde{SL_2}(\kappa,\tau)$. $S^3(\kappa,\tau)$ does not admit any isotropic geodesic. Finally, the tangent conjugate and isotropic conjugate locus of $M^3(\kappa,\tau)$ are given as union of surfaces of revolution about the Hopf direction.

2. Preliminaries

Let (M,g) be a connected homogeneous *Riemannian* manifold. As it is well-known, (M,g) can be expressed as coset space G/K, where G is a connected Lie group of isometries acting transitively and effectively on M, K is the isotropy subgroup of G at some point $o\in M$, the *origin* of M, and g is considered as a G-invariant Riemannian metric on G/K. Moreover, we can assume that G/K is a *reductive homogeneous space*, i.e., there is an $Ad(K)$-invariant subspace $\mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G such that $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{k}$, $\mathfrak{k}$ being the Lie algebra of K. Such quotient representation of (M,g) is in general not unique. $(M=G/K,g)$ is said to be *naturally reductive*, or more precisely *G-naturally reductive*, if there exists a reductive

decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ satisfying

$$< [X,Y]_{\mathfrak{m}}, Z > + < [X,Z]_{\mathfrak{m}}, Y >= 0 \tag{2.1}$$

for all $X, Y, Z \in \mathfrak{m}$, where $[X,Y]_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of $[X,Y]$ and $<,>$ is the metric induced by g on $\mathfrak{m}$, by using the canonical identification $\mathfrak{m} \cong T_oM$. If there exists a bi-invariant metric B on $\mathfrak{g}$ whose restriction to $\mathfrak{m} = \mathfrak{k}^{\perp}$ is the metric $<,>$, the (naturally reductive) space $(M = G/K, g)$ is called *normal homogeneous*. Then, for all $X, Y, Z \in \mathfrak{g}$, we have

$$B([X,Y],Z) + B([X,Z],Y) = 0. \tag{2.2}$$

For each $X \in \mathfrak{g}$, the mapping $\psi : I\!\!R \times M \to M$, $(t,p) \in I\!\!R \times M \mapsto \psi_t(p) = (\exp tX)p$ is a one-parameter group of isometries and consequently, ψ induces a Killing vector field X^* given by

$$X^*_p = \frac{d}{dt}_{|t=0}(\exp tX)p, \qquad p \in M. \tag{2.3}$$

X^* is called the *fundamental vector field* or the *infinitesimal G-motion* corresponding to X on M. If $G = I_o(M,g)$, then all (complete) Killing vector field on M is a fundamental vector field X^*, for some $X \in \mathfrak{g}$. For any $a \in G$, we have

$$(Ad_aX)^*_{ap} = a_{*p}X^*_p, \tag{2.4}$$

where a_{*p} denotes the differential map of a at $p \in M$.

Next, let $\tilde{T}$ denote the torsion tensor and $\tilde{R}$ the corresponding curvature tensor of the *canonical connection* $\tilde{\nabla}$ of (M,g) adapted to the reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ [13, I, p.110] defined by the sign convention $\tilde{R}(X,Y) = \tilde{\nabla}_{[X,Y]} - [\tilde{\nabla}_X, \tilde{\nabla}_Y]$ and $\tilde{T}(X,Y) = \tilde{\nabla}_XY - \tilde{\nabla}_YX - [X,Y]$, for all $X, Y \in \mathfrak{X}(M)$, the Lie algebra of smooth vector fields on M. Then, these tensors are given by

$$\tilde{T}_o(X,Y) = -[X,Y]_{\mathfrak{m}} \quad , \quad \tilde{R}_o(X,Y) = \mathrm{ad}_{[X,Y]_{\mathfrak{k}}} \tag{2.5}$$

and we have $\tilde{\nabla}g = \tilde{\nabla}\tilde{T} = \tilde{\nabla}\tilde{R} = 0$. On naturally reductive homogeneous manifolds $(M = G/K, g)$, the tensor field $S = \nabla - \tilde{\nabla}$, where ∇ denotes the Levi Civita connection of (M,g), is a *homogeneous structure* [20] satisfying $S_XY = -S_YX = -\frac{1}{2}\tilde{T}(X,Y)$, for all $X, Y \in \mathfrak{X}(M)$, and we get

$$\tilde{R}_{XY} = R_{XY} + [S_X, S_Y] - 2S_{S_XY}. \tag{2.6}$$

Then ∇ and $\tilde{\nabla}$ have the same geodesics and, consequently, the same Jacobi fields (see [21]). Such geodesics are orbits of one-parameter subgroups of G of type $\exp tu$ where $u \in \mathfrak{m}$. In what follows, we shall denote by γ_u the unit-speed geodesic starting at the origin o with $\gamma'_u(0) = u$, $\|u\| = 1$. Then $\gamma_u(t) := (\exp tu)o$ and the Jacobi equation for ∇ coincides with the Jacobi equation for $\tilde{\nabla}$,

$$\frac{\tilde{\nabla}^2V}{dt^2} - \tilde{T}_\gamma\frac{\tilde{\nabla}V}{dt} + \tilde{R}_\gamma V = 0,$$

where $\tilde{R}_\gamma = \tilde{R}(\gamma', \cdot)\gamma'$ and $\tilde{T}_\gamma = \tilde{T}(\gamma', \cdot)$. Taking into account that $\tilde{\nabla}\tilde{T} = \tilde{\nabla}\tilde{R} = 0$ and the parallel translation with respect to $\tilde{\nabla}$ of tangent vectors at the origin along γ_u coincides with

the differential of $\exp tu \in G$ acting on M, it follows that any Jacobi field V along $\gamma(t)$ can be expressed as $V(t) = (\exp tu)_{*o}X(t)$ where $X(t)$ is solution of the differential equation

$$X''(t) - \tilde{T}_u X'(t) + \tilde{R}_u X(t) = 0 \tag{2.7}$$

in the vector space $\mathfrak{m}$, being $\tilde{T}_u X = \tilde{T}(u,X) = -[u,X]_{\mathfrak{m}}$ and $\tilde{R}_u X = \tilde{R}(u,X)u = [[u,X]_{\mathfrak{k}}, u]$ (see [12], [21] for more details).

A Jacobi field V along γ_u with $V(0) = 0$ is said to be *G-isotropic*, or simply *isotropic* if $G = I_o(M,g)$, if and only if there exists $A \in \mathfrak{k}$ such that $V = A^* \circ \gamma$, or equivalently, if there exists an $A \in \mathfrak{k}$ such that (see [12])

$$V'(0) = [A,u]. \tag{2.8}$$

3. Jacobi osculating rank

Let $(M = G/K, g)$ be a connected naturally reductive homogeneous Riemannian manifold with adapted decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$. For each $t \in \mathbb{R}$, denote by $R_u(t)$ the $(1,1)$-tensor on $\mathfrak{m}$ obtained by the parallel translation of the Jacobi operator R_{γ_u} along the geodesic γ_u, i.e.,

$$R_u(t) = \tau_u^{-t} \circ R_{\gamma_u} \circ \tau_u^t,$$

where $\tau_u^t : T_oM \cong \mathfrak{m} \to T_{\gamma_u(t)}M$ is the parallel translation with respect to ∇ along γ_u from $o = \gamma_u(0)$ to $\gamma_u(t)$. Then $R_u(t)$ is a curve in the $n(n+1)/2$-dimensional vector space $\mathcal{S}(\mathfrak{m})$ of all self-adjoint operators of $(\mathfrak{m}, <,>)$ with $R_u(0) = R_u$, being $R_u := R(u,\cdot)u$. $\mathcal{S}(\mathfrak{m})$ is a Euclidean vector space with inner product defined by

$$< K, K' >= \sum_{i=1}^{n} < K(e_i)K'(e_i) >, \tag{3.9}$$

for all $K, K' \in \mathcal{S}(\mathfrak{m})$, where $\{e_1, \dots, e_n\}$ is an arbitrary orthonormal basis of $(\mathfrak{m}, <,>)$. Because, for naturally reductive spaces, the *connection function* $\Lambda : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ associated to ∇ is given by $\Lambda_x(y) = 1/2[x,y]_{\mathfrak{m}} = S_x y$, for all $x, y \in \mathfrak{m}$, [13, Theorem 2.10, p.197] it directly follows

$$\frac{\nabla(\exp tu)_{*o}v}{dt}_{|t} = (\exp tu)_{*o} S_u v,$$

for each $v \in \mathfrak{m}$, and hence the parallel translation τ_u is given by

$$\tau_u^t = (\exp tu)_{*o} e^{-tS_u}, \tag{3.10}$$

where e denotes the exponential map of the Lie group of the automorphisms $\mathrm{Aut}(\mathfrak{m})$ of $\mathfrak{m}$. Note that $S_u : \mathfrak{m} \to \mathfrak{m}$ is a skew-symmetric endomorphism of $(\mathfrak{m}, <,>)$ and so, e^{S_u} is a linear isometry of $(\mathfrak{m}, <,>)$.

Lemma 3.1. *We have:*

$$R_u(t) = Ad_{e^{tS_u}} R_u. \tag{3.11}$$

Proof. For $x, y \in \mathfrak{m}$, using (3.10) one obtains

$$\begin{aligned} < R_u(t)x, y > &= g_{\gamma_u(t)}(R_{\gamma_u(t)}\tau_u^t x, \tau_u^t y) = g_{\gamma_u(t)}(R_{\gamma_u(t)}(\exp tu)_{*o}e^{-tS_u}x, (\exp tu)_{*o}e^{-tS_u}y) \\ &= g_{\gamma_u(t)}((\exp tu)_{*o}R_u e^{-tS_u}x, (\exp tu)_{*o}e^{-tS_u}y) =< R_u e^{-tS_u}x, e^{-tS_u}y > \\ &= < e^{tS_u}\circ R_u \circ e^{-tS_u}(x), y >=< (Ad_{e^{tS_u}}R_u)x, y > . \end{aligned}$$

Then, we have proved (3.11). □

Hence, the curve $R_u(t)$ in $\mathcal{S}(\mathfrak{m})$ can be expressed as the power series expression

$$R_u(t) = e^{t\mathrm{ad}S_u}(R_u) = \sum_{k=0}^{\infty}\frac{t^k}{k!}S_u^k \cdot R_u, \tag{3.12}$$

where S_u acts as a derivation on the space of the endomorphisms $\mathrm{End}(\mathfrak{m})$ of $\mathfrak{m}$. It proves the following.

Lemma 3.2. *We have:*

$$R_u^{i)}(0) = S_u^i \cdot R_u, \quad \textit{for all } i \in \mathbb{N}.$$

A curve $\alpha : I \to M$ in an arbitrary manifold M is said that has *constant osculating rank* r if for all $t \in I$, its higher order derivatives $\alpha'(t), \dots, \alpha^{r)}(t)$ are linearly independent and $\alpha'(t), \dots, \alpha^{r+1)}(t)$ are linearly dependent in $T_{\alpha(t)}M$. Orbits of one-parameter subgroups of a Lie group acting on M are examples of curves with constant osculating rank. Concretely, we have

Lemma 3.3. *Let G be a Lie group acting on M from the left and let α be the curve in M given by $\alpha(t) = (\exp tX)p$, for some $X \in \mathfrak{g}$ and $p \in M$. Then,*

$$\alpha^{k)}(t) = (\exp tX)_{*p}\alpha^{k)}(0),$$

for all $t \in I\!R$ and $k \in \mathbb{N}$, and so it has constant osculating rank.

Proof. From (2.4), taking into account that α is the integral curve through p of the fundamental vector field X^*, we obtain $\alpha'(t) = (\exp tX)_{*p}\alpha'(0)$. Hence, it follows that $\alpha'(t+s) = (\exp tX)_{*\alpha(s)}\alpha'(s)$ and then,

$$\begin{aligned} \alpha''(t) &= \tfrac{d}{dt}_{|t}(\exp tX)_{*p}\alpha'(0) = \tfrac{d}{dt}_{|t}\tfrac{d}{ds}_{|s=0}(\exp(t+s)X)p = \tfrac{d}{ds}_{|s=0}\tfrac{d}{dt}_{|t}\alpha(t+s) \\ &= \tfrac{d}{ds}_{|s=0}\alpha'(t+s) = \tfrac{d}{ds}_{|s=0}(\exp tX)_{*\alpha(s)}\alpha'(s) = (\exp tX)_{*p}\alpha''(0). \end{aligned}$$

For the general case, we just use induction. Suppose that $\alpha^{i)}(t) = (\exp tX)_{*p}\alpha^{i)}(0)$, for any $i \le k-1$ and $k \ge 3$. We obtain

$$\begin{aligned} \alpha^{k)}(t) &= \tfrac{d}{dt}_{|t}(\exp tX)_{*p}\alpha^{k-1)}(0) = \tfrac{d}{dt}_{|t}\tfrac{d}{ds}_{|s=0}(\exp tX)_{*\alpha(s)}(\exp sX)_{*p}\alpha^{k-2)}(0) \\ &= \tfrac{d}{ds}_{|s=0}\tfrac{d}{dt}_{|t}(\exp(t+s)X)_{*p}\alpha^{k-2)}(0) = \tfrac{d}{ds}_{|s=0}\tfrac{d}{dt}_{|t}\alpha^{k-2)}(t+s) = \tfrac{d}{ds}_{|s=0}\alpha^{k-1)}(t+s) \\ &= \tfrac{d}{ds}_{|s=0}(\exp tX)_{*\alpha(s)}\alpha^{k-1)}(s) = (\exp tX)_{*p}\alpha^{k)}(0). \end{aligned}$$

□

Because $\{Ad_{e^{tS_u}} \mid t \in I\!R\}$ is a one-parameter subgroup of the isometry group of $(\mathcal{S}(\mathfrak{m}), <,>)$, it follows from Lemmas 3.1 and 3.3 that it has constant osculating rank, the *Jacobi osculating rank of* γ_u. For each unit vector $u \in \mathfrak{m}$, denote by $\mathcal{R}_u(\mathfrak{m})$ the smallest subspace of $\mathcal{S}(\mathfrak{m})$ such that R_u and $S_u \cdot \mathcal{R}_u(\mathfrak{m}) \subset \mathcal{R}_u(\mathfrak{m})$. From Lemma 3.2, $\mathcal{R}_u(\mathfrak{m})$ is generated by R_u,

$S_u \cdot R_u, \ldots, S_u^r \cdot R_u$. Hence, $r = \dim \mathcal{R}_u(\mathfrak{m}) - 1$ or $r = \dim \mathcal{R}_u(\mathfrak{m})$ and then, taking into account that $K(u) = 0$, for each $K \in \mathcal{R}_u(\mathfrak{m})$, we have

$$\mathrm{rank}_{\mathrm{osc}}(u) \leq \dim \mathcal{R}_u(\mathfrak{m}) \leq \frac{n(n-1)}{2}.$$

Remark 3.4. A homogeneous Riemannian manifold $(M = G/K, g)$ is said to be a *g.o. space* if each geodesic starting at the origin is an orbit of an one-parameter subgroup $(\exp tZ)$ of $Z \in \mathfrak{g}$. Naturally reductive spaces are g.o. spaces but there is a large number of examples of g.o. spaces which are not naturally reductive. In similar way as before, one obtains that the parallel translation τ_u along the geodesic $\gamma_u(t) = (\exp tZ)o$ on a g.o. space is given by

$$\tau_u^t = (\exp tZ)_{*o} e^{-t\Lambda_u}$$

and then the formulas (3.11), (3.12) and Lemma 3.2 hold. Hence, the notion of Jacobi osculating rank can be directly extended to g.o. spaces (see [1] for more details and references).

When the Jacobi osculating rank does not depend on the choice of the geodesic, the naturally reductive homogeneous manifold (M, g) is said to have *constant Jacobi osculating rank.*

Lemma 3.5. *Any naturally reductive homogeneous manifold* (M, g) *of constant Jacobi osculating rank zero is locally symmetric.*

Proof. For each unit vector u in $\mathfrak{m}$, let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues and $\{e_1, \ldots, e_n\}$ the corresponding eigenvectors of the operator R_u. Because $\mathrm{rank}_{\mathrm{osc}}(u) = 0$, it follows that $R_u(t) = R_u$, for all $t \in I\!R$, and then, $\tau_u \circ R_u = R_{\gamma_u} \circ \tau_u$. Hence, $R_{\gamma_u} E_i = \lambda_i E_i$, $i = 1, \ldots, n$, where E_i is the vector field along γ_u given by $E_i(t) = \tau_u^t(e_i)$. Then $\{E_i, \ldots, E_n\}$ becomes into a parallel frame field of eigenvectors of the Jacobi operator R_{γ_u} with constant eigenvalues λ_i and so, (M, g) must be locally symmetric [2]. $\square$

Because symmetric spaces have constant Jacobi osculating rank zero, then we have

Theorem 3.6. *A simply connected, naturally reductive homogeneous manifold has constant Jacobi osculating rank zero if and only if it is a symmetric space.*

Next, we also consider the curve $\tilde{R}_u(t)$ in $\mathcal{S}(\mathfrak{m})$ obtained by the ∇-parallel translation of $\tilde{R}_{\gamma_u}$ along γ_u. First, using (2.6), we get

$$\tilde{R}_{\gamma_u} = R_{\gamma_u} + S_{\gamma_u}^2. \tag{3.13}$$

Lemma 3.7. *The curves* $R_u(t)$ *and* $\tilde{R}_u(t)$ *have the same osculating rank. Moreover, we have:*

(i) $\tilde{R}_u(t) = R_u(t) + S_u^2$.
(ii) $R_u^{i)}(0) = \tilde{R}_u^{i)}(0) = S_u^i \cdot \tilde{R}_u$, *for all* $i \in \mathbb{N}$.

Proof. Because $\frac{\nabla}{dt} = \frac{\tilde{\nabla}}{dt} + S_{\gamma_u}$, one obtains $\frac{\nabla S_{\gamma_u}}{dt} = \frac{\tilde{\nabla} S_{\gamma_u}}{dt}$ and, using that $\tilde{\nabla} S = 0$, the tensor field S_{γ_u} is ∇-parallel along γ_u, or equivalently, $\tau_u^t \circ S_u = S_{\gamma_u} \circ \tau_u^t$. Then, the curve $S_u(t) = \tau_u^{-t} \circ S_{\gamma_u} \circ \tau_u^t$ is constant and (i) follows by using (3.13). Now, using Lemma 3.2, we also get (ii). $\square$

On naturally reductive homogeneous spaces $(M = G/K, g)$, any G-invariant (unit) vector field is Killing and so, each one of its integral curves is a geodesic.

Lemma 3.8. *The Jacobi osculating rank of each integral curve of a* G*-invariant vector field is zero.*

Proof. Put $u = U_o \in \mathfrak{m}$. Then u is $Ad(K)$-invariant and it implies $[\mathfrak{k}, u] = 0$ and from (2.5), $\tilde{R}_u = 0$. Because $\tilde{R}$ is G-invariant, it follows that $\tilde{R}_{\gamma_u} \circ (\exp tu)_{*o} = (\exp tu)_{*o} \circ \tilde{R}_u$ and then $\tilde{R}_{\gamma_u}$ vanishes along γ_u. From Lemma 3.7, $\mathrm{rank}_{\mathrm{osc}}(u) = 0$. $\square$

Hence, using Lemma 3.5, we have

Proposition 3.9. *Any non-locally symmetric naturally reductive space $(M = G/K, g)$ with a G-invariant vector field has non-constant Jacobi osculating rank.*

In [12, Lemma 5.5] it is proved that any non-locally symmetric naturally reductive space (M, g) of dimension $n \leq 5$ admits a naturally reductive quotient representation G/K and a (non-parallel) G-invariant unit vector field. Then, we can conclude

Corollary 3.10. *Any non-locally symmetric naturally reductive space of dimension $n \leq 5$ has non-constant Jacobi osculating rank.*

Remark 3.11. Simply connected Killing-transversally symmetric spaces are introduced in [10] as simply connected Riemannian manifolds equipped with a complete unit Killing vector field ξ such that all reflections with respect to its integral curves are isometries. This family of naturally reductive spaces contains to $M^3(\kappa, \tau)$, for all κ and $\tau > 0$, and ξ is G-invariant with respect to a naturally reductive representation $M = G/K$. The dual one-form of ξ with respect to the metric is a contact form if and only if it is irreducible [10, Theorem 5.1]. Then we can give a lot of examples of irreducible naturally reductive spaces with non-constant Jacobi osculating rank: Generalized Heisenberg groups equipped with suitable left-invariant metrics, Berger's spheres, φ-symmetric spaces, Sasakian space forms, etc.

4. Jacobi osculating rank of geodesics on $M^3(\kappa, \tau)$

A naturally reductive decomposition for the Lie algebra of the isometry group and an adapted canonical connection for $M^3(\kappa, \tau)$ are obtained in [20] by using of naturally reductive homogeneous structures. Next, we give a brief summary from Theorems 6.4 and 6.5 in [20]. A non-vanishing naturally reductive homogeneous structure on an arbitrary three-dimensional (oriented) Riemannian manifold (M^3, g) can be expressed as $S = \lambda dv$, for some non-zero constant λ, where dv is the volume form on M^3 and S also denotes the 3-form $S(X, Y, Z) = g(S_X Y, Z)$, for all $X, Y, Z \in \mathfrak{X}(M^3)$. Because S and $-S$ are isomorphic structures, we can take $\lambda > 0$. Then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{m} \cong T_o M^3$ such that

$$S_{e_1} e_2 = \lambda e_3, \quad S_{e_3} e_1 = \lambda e_2, \quad S_{e_2} e_3 = \lambda e_1. \tag{4.14}$$

For $M^3(\kappa, \tau)$, putting $e_3 = \xi_o$, we get $\lambda = \frac{\tau}{2}$ and $\tilde{R}$ is expressed as

$$\tilde{R}_{e_1 e_2} = (\kappa - \tau^2) A_{12}, \quad \tilde{R}_{e_1 e_3} = 0, \quad \tilde{R}_{e_2 e_3} = 0, \tag{4.15}$$

where A_{12} is the skew-symmetric endomorphism on $\mathfrak{m}$ given by

$$A_{12} e_1 = e_2, \quad A_{12} e_2 = -e_1, \quad A_{12} e_3 = 0.$$

Note that if $\kappa = \tau^2$, then $\tilde{R}$ vanishes and $M^3(\kappa, \tau)$ is an Einstein manifold and hence, of constant curvature. The holonomy algebra $\mathfrak{k}$ of $\tilde{\nabla}$ is generated by A_{12} and the transvection

algebra $\mathfrak{tr}(\mathfrak{m}) = \mathfrak{m} \oplus \mathfrak{k}$ is generated by $\{e_1, e_2, e_3, A_{12}\}$ with Lie bracket given by

$$\left\{ \begin{array}{lll} [e_1, e_2] = \tau e_3 + (\kappa - \tau^2) A_{12}, & [e_1, e_3] = -\tau e_2, & [e_2, e_3] = \tau e_1 \\ [A_{12}, e_1] = e_2, & [A_{12}, e_2] = -e_1, & [A_{12}, e_3] = 0. \end{array} \right. \tag{4.16}$$

Hence, putting

$$u_1 = e_1, \quad u_2 = e_2, \quad u_3 = e_3 + \frac{\kappa - \tau^2}{\tau} A_{12}, \tag{4.17}$$

the subspace $\mathfrak{h}$ of $\mathfrak{tr}(\mathfrak{m})$ generated by u_1, u_2, u_3 is a three-dimensional unimodular Lie algebra and the linear isometry $f : T_o M^3(\kappa, \tau) \cong \mathfrak{m} \to \mathfrak{h}$ given by $f(e_i) = u_i$, $i = 1, 2, 3$, determines a (unique) isometry from $M^3(\kappa, \tau)$ to the connected and simply connected Lie group with Lie algebra $\mathfrak{h}$ equipped with the left invariant metric such that $\{u_1, u_2, u_3\}$ is an orthonormal basis. Then we can identify $M^3(\kappa, \tau)$ with this unimodular Lie group and u_3 with the Hopf vector field ξ. From (4.17), we get

$$\begin{array}{llllllll} [u_1, u_2] & = & \tau u_3, & [u_2, u_3] & = & \frac{\kappa}{\tau} u_1, & [u_3, u_1] & = & \frac{\kappa}{\tau} u_2, \\ [A_{12}, u_1] & = & u_2, & [A_{12}, u_2] & = & -u_1, & [A_{12}, u_3] & = & 0. \end{array} \tag{4.18}$$

Hence, it follows that $\mathfrak{tr}(\mathfrak{m})$ is a semi-direct sum $\mathfrak{h} \times_\beta \mathfrak{k}$ of $\mathfrak{h}$ and $\mathfrak{k}$ where β is the homomorphism of $\mathfrak{k}$ into $End(\mathfrak{h})$ given by $\beta(A_{12})u_1 = u_2$, $\beta(A_{12})u_2 = -u_1$, $\beta(A_{12})u_3 = 0$ and we have

$$\nabla_X \xi = \frac{\tau}{2} X \times \xi,$$

where $\times$ denotes the vector product in $\mathfrak{h}$. It implies that ξ is a unit Killing vector field and so, the geodesic $\gamma_{e_3}(t) = (\exp t e_3) o$ coincides with its integral curve through the origin. Moreover, the sectional curvature $K(X, \xi)$ of the two-plane spanned by X, ξ is a non-negative constant c^2, called the *ξ-sectional curvature* [10], given by $c^2 = \frac{\tau^2}{4}$ and $\{u_1, u_2, \xi\}$ is in fact a basis of eigenvectors for the Ricci tensor ρ with $\rho(u_1, u_1) = \rho(u_2, u_2) = \kappa - \frac{\tau^2}{2}$ and $\rho(\xi, \xi) = \frac{\tau^2}{2}$. Furthermore, ξ is the vertical field of the fibration $M^3(\kappa, \tau) \to M^3/\xi = M^2(\kappa)$. Then $M^3(\kappa, \tau)$ is a principal G^1-bundle, where G^1 denotes the one-parameter subgroup of global isometries φ_t generated by ξ. G^1 is isomorphic to either the circle group S^1 or to $\mathbb{R}$ depending on whether the integral curves of ξ are closed or not. In particular, G^1 is a circle when $M^3(\kappa, \tau)$ is compact. If $c = 0$, or equivalently τ vanishes, this fibration becomes trivial and we get the product spaces $M^2(\kappa) \times \mathbb{R}$. For $\kappa > 0$, the metrics $g = g_{\kappa,\tau}$ are known as *Berger metrics* and the corresponding fibration, as the *Hopf fibration*. Here, as it has been already said, ξ is called the *Hopf vector field* of $M^3(\kappa, \tau)$, even for the cases $\kappa \leq 0$.

We shall use the following result, which can be of interest by itself.

Lemma 4.1. *Each non-symmetric connected and simply connected three-dimensional normal homogeneous manifold is isometric to $S^3(\kappa, \tau)$, for some pair (κ, τ) such that $\kappa > \tau^2$.*

Proof. The set of all inner products on $\mathfrak{tr}(\mathfrak{m})$ whose restriction to $\mathfrak{m}$ is $<,>$ and such that $\mathfrak{k}$ is orthogonal to $\mathfrak{m}$ form an one-parameter family $\{B_r \mid r > 0\}$, where $B_r(A_{12}, A_{12}) = r$. Then, from (2.2) and using (4.16), B_r is bi-invariant if and only if $r = \frac{1}{\kappa - \tau^2}$. So, the existence of bi-invariant metrics on $M^3(\kappa, \tau)$ is determined by the condition $\kappa - \tau^2 > 0$. $\square$

Each $u \in \mathfrak{m}$ can be written as $u(\theta,\phi) = \sin\theta\cos\phi e_1 + \sin\theta\sin\phi e_2 + \cos\theta e_3$, where $\theta \in [0,\pi]$ and $\phi \in [0, 2\pi]$. Then, we get

$$u(\theta,\phi) = e^{\phi A_{12}} u(\theta), \tag{4.19}$$

where $u(\theta)$ denotes the unit vector in the plane $I\!R\{e_1, e_3\}$ given by $u(\theta) = u(\theta, 0)$.

Lemma 4.2. *We have:*

(a) $(\exp\phi A_{12})\gamma_{u(\theta)} = \gamma_{u(\theta,\phi)}$.
(b) $S_{u(\theta,\phi)} = Ad_{e^{\phi A_{12}}} S_{u(\theta)}$, $\tilde{R}_{u(\theta,\phi)} = Ad_{e^{\phi A_{12}}} \tilde{R}_{u(\theta)}$, $R_{u(\theta,\phi)} = Ad_{e^{\phi A_{12}}} R_{u(\theta)}$.
(c) $R_{u(\theta,\phi)}(t) = Ad_{e^{\phi A_{12}}} R_{u(\theta)}(t)$.
(d) $\mathrm{rank}_{\mathrm{osc}} u(\theta,\phi) = \mathrm{rank}_{\mathrm{osc}} u(\theta)$, *for all* $\phi \in [0, 2\pi]$.

Proof. From (4.16), we get $Ad_{\exp\phi A_{12}} = e^{\phi ad A_{12}} = e^{\phi A_{12}}$. Then (4.19) can be written as

$$u(\theta,\phi) = Ad_{\exp\phi A_{12}} u(\theta).$$

Hence, (a) follows taking into account that the action of the linear isotropy group of the isotropy subgroup K of $M^3(\kappa,\tau)$ at the origin corresponds under projection with $Ad(K)$ on $\mathfrak{m}$. Note that K is connected because $M^3(\kappa,\tau)$ is simply connected and then it coincides with the one-parameter subgroup $\exp\phi A_{12}$.

In (b), we shall only prove the first equality, the other two are obtained in similar way. Because the tensor field S is invariant for the isometry group of $M^3(\kappa,\tau)$, S as tensor on $\mathfrak{m}$ is $Ad(K)$-invariant and then $S_{u(\theta,\phi)} \circ Ad_{\exp\phi A_{12}} = Ad_{\exp\phi A_{12}} \circ S_{u(\theta)}$. Hence, we obtain

$$S_{u(\theta,\phi)} = Ad_{Ad_{\exp\phi A_{12}}} S_{u(\theta)} = Ad_{e^{\phi A_{12}}} S_{u(\theta)}.$$

For (c), we use Lemma 3.2 and (b) to obtain

$$R^{i)}_{u(\theta,\phi)}(0) = Ad_{e^{\phi A_{12}}} R^{i)}_{u(\theta)}(0). \tag{4.20}$$

Finally, because $Ad_{e^{\phi A_{12}}}$ belongs to $\mathrm{Aut}(\mathcal{S}(\mathfrak{m}))$, we have (d). □

Therefore, we may restrict our study of the geometry of geodesics on $M^3(\kappa,\tau)$ to geodesics emanating from the origin with initial directions $u(\theta)$, $\theta \in [0,\pi]$.

Theorem 4.3. *The Jacobi osculating rank of every geodesic γ_u on $M^3(\kappa,\tau)$ is two except for the Hopf fibers, where it is zero. Moreover, we have:*

(i) $R_u(t) = R_u + \frac{1}{\tau}\Big(\sin\tau t R'_u(0) + \frac{1}{\tau}(1-\cos\tau t)R''_u(0)\Big)$.
(ii) *For each $u = u(\theta,\phi)$ with $\theta \in]0,\pi[$, $R_u(t)$ is a circle in $\mathcal{S}(\mathfrak{m})$ of radius* $\frac{\sqrt{2}}{2}|\tau^2-\kappa|\sin^2\theta$.

Proof. With respect to the basis $\{e_1, e_2, e_3\}$ on $\mathfrak{m}$, using (4.14) and (4.15), we get

$$S_{u(\theta)} = \frac{\tau}{2}\begin{pmatrix} 0 & -\cos\theta & 0 \\ \cos\theta & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix}; \quad \tilde{R}_{u(\theta)} = 2\tau^2\mu(\theta)\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\mu(\theta) = \frac{\tau^2-\kappa}{2\tau^2}\sin^2\theta$. Note that $\mu(\theta) = 0$ if and only if $\theta = 0$ or $\theta = \pi$, i.e. on the Hopf direction. By a direct computation, we have

$$S_{u(\theta)} \cdot \tilde{R}_{u(\theta)} = \tau^3\mu(\theta)\begin{pmatrix} 0 & \cos\theta & 0 \\ \cos\theta & 0 & -\sin\theta \\ 0 & -\sin\theta & 0 \end{pmatrix}$$

and

$$S^2_{u(\theta)} \cdot \tilde{R}_{u(\theta)} = \tau^4\mu(\theta)\begin{pmatrix} -\cos^2\theta & 0 & \sin\theta\cos\theta \\ 0 & 1 & 0 \\ \sin\theta\cos\theta & 0 & -\sin^2\theta \end{pmatrix}.$$

Moreover, one obtains $S^3_{u(\theta)} \cdot \tilde{R}_{u(\theta)} = -\tau^2 S_{u(\theta)} \cdot \tilde{R}_{u(\theta)}$ and $S^4_{u(\theta)} \cdot \tilde{R}_{u(\theta)} = -\tau^2 S^2_{u(\theta)} \cdot \tilde{R}_{u(\theta)}$. Therefore, taking into account that $S^{k+1} \cdot \tilde{R}_{u(\theta)} = S_{u(\theta)} \circ (S^k \cdot \tilde{R}_{u(\theta)}) - (S^k \cdot \tilde{R}_{u(\theta)}) \circ S_{u(\theta)}$, it follows by the induction

$$\begin{aligned} S^{2k-1}_{u(\theta)} \cdot \tilde{R}_{u(\theta)} &= (-1)^{k-1}\tau^{2(k-1)} S_{u(\theta)} \cdot \tilde{R}_{u(\theta)}, \\ S^{2k}_{u(\theta)} \cdot \tilde{R}_{u(\theta)} &= (-1)^{k-1}\tau^{2(k-1)} S^2_{u(\theta)} \cdot \tilde{R}_{u(\theta)}. \end{aligned} \tag{4.21}$$

Then, from Lemma 3.7 and Lemma 4.2 (d), we obtain that $\mathrm{rank}_{\mathrm{osc}} u(\theta,\phi) = 2$ if $\theta \in]0,\pi[$ and $\mathrm{rank}_{\mathrm{osc}}(\pm e_3) = 0$ and it proves the first part of the Theorem. Moreover, (i) follows from (4.21), using Lemma 3.7, Lemma 4.2 (b) and (4.20).

For $\theta \in]0,\pi[$, the elements $\{v_1(\theta), v_2(\theta), v_3(\theta)\}$ of $\mathcal{S}(\mathfrak{m})$ given by the matrices

$$v_1(\theta) = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & \cos\theta & 0 \\ \cos\theta & 0 & -\sin\theta \\ 0 & -\sin\theta & 0 \end{pmatrix}, \quad v_2(\theta) = \frac{\sqrt{2}}{2}\begin{pmatrix} -\cos^2\theta & 0 & \sin\theta\cos\theta \\ 0 & 1 & 0 \\ \sin\theta\cos\theta & 0 & -\sin^2\theta \end{pmatrix},$$

$$v_3(\theta) = \frac{\sqrt{2}}{2}\begin{pmatrix} -\cos^2\theta & 0 & \sin\theta\cos\theta \\ 0 & -1 & 0 \\ \sin\theta\cos\theta & 0 & -\sin^2\theta \end{pmatrix}$$

constitute an orthonormal basis for $R_{u(\theta)}(\mathfrak{m})$ in $(\mathcal{S}(\mathfrak{m}), <,>)$ and we get

$$\begin{aligned} \tilde{R}_{u(\theta)} &= \sqrt{2}\tau^2\mu(\theta)(v_3(\theta) - v_2(\theta)), & S^2_{u(\theta)} &= \tfrac{\sqrt{2}}{4}\tau^2 v_3(\theta), \\ R'_{u(\theta)}(0) &= \sqrt{2}\tau^3\mu(\theta)v_1(\theta), & R''_{u(\theta)}(0) &= \sqrt{2}\tau^4\mu(\theta)v_2(\theta), \\ R_{u(\theta)} &= -\tfrac{\sqrt{2}}{4}\tau^2\Big(4\mu(\theta)v_2(\theta) + (1-4\mu(\theta))v_3(\theta)\Big). \end{aligned} \tag{4.22}$$

Hence, (i) implies that

$$R_{u(\theta)}(t) = \sqrt{2}\tau^2\mu(\theta)\Big(\sin\tau t v_1(\theta) - \cos\tau t v_2(\theta)\Big) + \frac{\sqrt{2}}{4}\tau^2(\mu(\theta) - 1)v_3(\theta).$$

Then, putting $R_{u(\theta)}(\mathfrak{m}) \cong I\!R^3[x,y,z]$, where x, y, z are the cartesian coordinates with respect to $\{v_1(\theta), v_2(\theta), v_3(\theta)\}$, $R_{u(\theta)}(t)$ is the circle in the plane $z = \frac{\sqrt{2}}{4}\tau^2(\mu(\theta)-1)$ such that $x = \sqrt{2}\tau^2\mu(\theta)\sin\tau t$, $y = -\sqrt{2}\tau^2\mu(\theta)\cos\tau t$. Because the Euclidean product $<,>$ of $\mathcal{S}(\mathfrak{m})$, defined

in (3.9), coincides with the trace form of End($\mathfrak{m}$), it is, in particular, $Ad(e^{\phi A_{12}})$-invariant and so, (ii) follows from Lemma 4.2 (c), taking into account that $S^2_{u(\theta,\phi)} = Ad_{e^{\phi A_{12}}} S^2_{u(\theta)}$. □

Remark 4.4. The circle $R_{u(\theta,\phi)}(t)$ is in the plane of $\mathcal{S}(\mathfrak{m})$ determined by its centre, i.e. the point $(4\mu(\theta)-1)S^2_{u(\theta,\phi)}$, and the subspace generated by $R'_{u(\theta,\phi)}(0)$ and $R''_{u(\theta,\phi)}(0)$; its period is $\frac{2\pi}{\tau}$ and it contracts to R_{ξ_0} when θ converges to 0 or to π.

5. Isotropic conjugate points in $M^3(\kappa,\tau)$

Next, we determine all pairs of conjugate points of $M^3(\kappa,\tau)$ and those which are isotropic or strictly isotropic. As before, we only need to consider geodesics $\gamma_{u(\theta)}$, with $\theta\in[0,\pi]$. Note that if $\gamma_{u(\theta)}$ is an isotropic geodesic, then $\gamma_{u(\theta,\phi)}$ is also isotropic. From (2.8), a Jacobi field V along $\gamma_{u(\theta)}$ in $M^3(\kappa,\tau)$, with $V(0)=0$ is isotropic if and only if $V'(0)$ is collinear with e_2. Hence, $\dim \mathrm{Isot}(\gamma_{u(\theta,\phi)})=1$ if $\theta\in]0,\pi[$ and zero for the fibers of $M^3(\kappa,\tau)$, where $\mathrm{Isot}(\gamma_{u(\theta,\phi)})$ denotes the space of all isotropic Jacobi fields along $\gamma_{u(\theta,\phi)}$ vanishing at the origin. Moreover, V along $\gamma_{u(\theta,\phi)}$ is isotropic if and only if $V'(0)$ is collinear with $e^{\phi A_{12}}e_2 = -\sin\phi e_1 + \cos\phi e_2$. Therefore, we also have

Lemma 5.1. *All pair of isotropic conjugate points in $M^3(\kappa,\tau)$ are strictly isotropic and all isotropic geodesic is strictly isotropic.*

Now, we prove the main theorem of this section.

Theorem 5.2. *A geodesic $\gamma(t)$ on $M^3(\kappa,\tau)$ starting at the origin with slope angle θ admits conjugate points to the origin if and only if $\lambda(\theta)>0$, where λ is the function given by $\lambda(\theta):=\kappa\sin^2\theta+\tau^2\cos^2\theta$. Moreover, we have:*

(i) *The conjugate points along the Hopf fibers are at $t=\frac{2\pi p}{\tau}$, $p\in\mathbb{N}$, their multiplicity is 2 and they are not isotropic.*

(ii) *For $\theta\in]0,\pi[$, the conjugate points along γ to the origin are all $\gamma(\frac{s}{\sqrt{\lambda(\theta)}})$, where*

  1. *$s=2p\pi$, $p\in\mathbb{N}$,*
  or
  2. *s is a solution of the equation $\tan\frac{s}{2}=\mu(\theta)s$, where $\mu(\theta)=\frac{\tau^2-\kappa}{2\tau^2}\sin^2\theta$.*

  In the first case, they are strictly isotropic and the second one, they are not isotropic. In both cases, their multiplicity is 1.

Proof. First, we shall show that the *tangent conjugate locus* $\mathrm{conj}(M^3(\kappa,\tau))$ of the origin is given by

$$\mathrm{conj}(M^3(\kappa,\tau)) = \{\frac{s}{\sqrt{\lambda(\theta)}}u(\theta,\phi) \mid \lambda(\theta)>0,\ s\in\mathcal{Z}^+(f_\theta)\},$$

where $\mathcal{Z}^+(f_\theta)$ is the set of zeros $s\in\mathbb{R}^+$ of $f_\theta(s)=1-\cos s-\mu(\theta)s\sin s$. (See *Fig. 1* and *Fig. 2*).

The Jacobi equation (2.7) for geodesics $\gamma_{u(\theta)}$ may be expressed, from (4.14) and (4.15), as

$$\left\{\begin{array}{l} X^{1''}-\tau\cos\theta X^{2'}=0,\\ X^{2''}+\tau(\cos\theta X^{1'}-\sin\theta X^{3'})-2\tau^2\mu(\theta)X^2=0,\\ X^{3''}+\tau\sin\theta X^{2'}=0, \end{array}\right. \tag{5.23}$$

where the solutions $X(t)$ in $\mathfrak{m}$ are given by $X(t) = \sum_{i=1}^{3} X^i(t)e_i$. If $\theta = 0$ or $\theta = \pi$, then it reduces to

$$\begin{cases} X^{1\prime\prime} - \tau X^{2\prime} = 0, \\ X^{2\prime\prime} + \tau X^{1\prime} = 0, \\ X^{3\prime\prime} = 0 \end{cases}$$

and the Jacobi solutions X such that $X(0) = 0$ are given by

$$X(t) = (A(1 - \cos\tau t) - B\sin\tau t)e_1 + (A\sin\tau t + B(1 - \cos\tau t)e_2 + Cte_3,$$

where A, B, C are constant. Hence, the conjugate points to the origin along the geodesic $\gamma(t) = (\exp te_3)o$ are given by $\gamma(2\pi p/\tau)$, for $p \in \mathbb{Z}^*$, the multiplicity of each one of them is 2 and they are not isotropic using (2.8). It proves (i).

Next, we suppose that $\theta \in]0, \pi[$. Differentiating the second equality and substituting in it the first and the second one, the system (5.23) can be reduced to

$$\begin{cases} X^{1\prime\prime} = \tau\cos\theta Y, \\ Y'' + \lambda Y = 0, \\ X^{3\prime\prime} = -\tau\sin\theta Y, \end{cases} \tag{5.24}$$

where $Y = X^{2\prime}$. If $\lambda = \lambda(\theta) > 0$, then $Y = A\cos\sqrt{\lambda}t + B\sin\sqrt{\lambda}t$, where A and B are constants. It is straightforward to check that the solutions $X^i(t)$ of above system such that $X^i(0) = 0$, $i = 1, 2, 3$, for $\theta \neq \frac{\pi}{2}$ are given by

$$\begin{array}{rcl} X^1(t) & = & \frac{\tau\cos\theta}{\lambda}\Big(A(1 - \cos\sqrt{\lambda}t) - B\sin\sqrt{\lambda}t + C\sqrt{\lambda}t\Big), \\ X^2(t) & = & \frac{1}{\sqrt{\lambda}}\Big(A\sin\sqrt{\lambda}t + B(1 - \cos\sqrt{\lambda}t)\Big), \\ X^3(t) & = & -\frac{\tau\sin\theta}{\lambda}\Big(A(1 - \cos\sqrt{\lambda}t) + B(\frac{\tau^2-\kappa}{\tau^2}\sqrt{\lambda}t - \sin\sqrt{\lambda}t) - C\cot^2\theta\sqrt{\lambda}t\Big). \end{array}$$

Then the arc length t, $t \neq 0$, at the conjugate points along γ are the zeros of the determinant

$$\begin{vmatrix} 1 - \cos\sqrt{\lambda}t & -\sin\sqrt{\lambda}t & 1 \\ \sin\sqrt{\lambda}t & 1 - \cos\sqrt{\lambda}t & 0 \\ 1 - \cos\sqrt{\lambda}t & \frac{\tau^2-\kappa}{\tau^2}\sqrt{\lambda}t - \sin\sqrt{\lambda}t & -\cot^2\theta \end{vmatrix}.$$

For $\theta = \frac{\pi}{2}$, the corresponding solutions are given by

$$\begin{array}{rcl} X^1(t) & = & Ct \\ X^2(t) & = & \frac{1}{\sqrt{\kappa}}\Big(A\sin\sqrt{\kappa}t + B(1 - \cos\sqrt{\kappa}t)\Big), \\ X^3(t) & = & -\frac{\tau}{\kappa}\Big(A(1 - \cos\sqrt{\kappa}t) + B(\frac{\tau^2-\kappa}{\tau^2}\sqrt{\kappa}t - \sin\sqrt{\kappa}t\Big). \end{array}$$

Hence, making straight calculations and putting $s = s(t) = \sqrt{\lambda}t$, one obtains that for both cases the problem to find the conjugate points to the origin, reduces to find the zeros of the function $f_\theta(s) = 1 - \cos s - \mu(\theta)s\sin s$, for $\theta \in [0, \pi]$. Hence, it follows that $\sin s = 0$, or equivalently $s \in 2\pi\mathbb{Z}$, or, putting $\mu = \mu(\theta)$,

$$\cos s = \frac{1 - \mu^2 s^2}{1 + \mu^2 s^2}, \qquad \sin s = \frac{2\mu s}{1 + \mu^2 s^2},$$

which yields to the equation $\tan\frac{s}{2} = \mu s$. Because the rank of these matrices is two, the multiplicity $n_{\gamma_{u(\theta)}}(s/\sqrt{\lambda(\theta)})$ of the conjugate point $\gamma_{u(\theta)}(s/\sqrt{\lambda(\theta)})$ is one. Moreover, the space of isotropic Jacobi fields along γ are spanned by $V(t) = (\exp tu)_{*o}X(t)$, where

$$X(t) = \tau\cos\theta(1-\cos\sqrt{\lambda}t)e_1 + \sqrt{\lambda}\sin\sqrt{\lambda}te_2 - \tau\sin\theta(1-\cos\sqrt{\lambda}t)e_3.$$

Therefore, the isotropic conjugate points to the origin are all $\gamma(t)$ with $t\sqrt{\lambda(\theta)} \in 2\pi\mathbb{Z}^*$.

For the rest of the cases, that is for $\lambda(\theta) \le 0$, one can check by straightforward computations that the geodesic $\gamma_{u(\theta)}$ does not admit conjugate points to the origin, which proves the first part of the theorem. We have also proved that $\mathcal{Z}^+(f_\theta) = \{2p\pi \mid p \in \mathbb{N}\}$, if $\theta = 0$ or $\theta = \pi$ and $\mathcal{Z}^+(f_\theta) = \{2p\pi \mid p \in \mathbb{N}\}\bigcup\{s \in I\!R \mid \tan\frac{s}{2} = \mu(\theta)s\}$ if $\theta \in]0,\pi[$. Hence, using Lemma 5.1, we get (i) and (ii). □

Next, we give some applications of Theorem 5.2. We start determining the isotropic geodesics of $M^3(\kappa,\tau)$.

5.1. **Isotropic geodesics.** As well-known, every maximal geodesic of a homogeneous Riemannian manifold is either one-to-one or simply closed.

Proposition 5.3. *We have:*

(i) *All closed geodesic on $M^3(\kappa,\tau)$ starting at the origin with slope angle $\theta \in]0,\pi[$ admits isotropic conjugate points and its length is a integer multiple of $\frac{2\pi}{\sqrt{\lambda(\theta)}}$.*

(ii) *Any isotropic geodesic is one-to-one.*

Proof. Let γ_u be a closed geodesic starting at the origin with length l and slope angle $\theta \in]0,\pi[$. Then the vector field $V = A^*_{12}\circ\gamma_u$ along γ_u satisfies $V(0) = V(l) = 0$. Moreover, it is an non zero isotropic Jacobi field. In fact, we have (see, for example [21, p. 577])

$$V'(0) = [A_{12}, u] = A_{12}(u) \neq 0.$$

Hence, $\gamma_u(0)$ and $\gamma_u(l)$ ($\gamma_u(0) = \gamma_u(l)$ is the origin of $M^3(\kappa,\tau)$) are isotropic conjugate points and, using Theorem 5.2, there exists $p \in \mathbb{N}$ such that $l = \frac{2p\pi}{\sqrt{\lambda(\theta)}}$. It proves (i). Because the existence of isotropic conjugate points to the origin along a geodesic implies also, using Theorem 5.2, the existence of non-isotropic conjugate points, (i) allows to show (ii). □

In [7, Corollary 3.8 and Proposition 3.10], S. Engel has proved that a geodesic on $M^3(\kappa,\tau)$ with slope angle $\theta \in]0,\pi[$, such that $\lambda(\theta) > 0$ intersects the Hopf fiber through the origin periodically at the points $\gamma(2p\pi/\sqrt{\lambda(\theta)})$, $p \in \mathbb{N}$. Then, using Theorem 5.2, we can show the following.

Proposition 5.4. *Any geodesic on $M^3(\kappa,\tau)$ starting at the origin with slope angle $\theta \in]0,\pi[$ intersects the Hopf fiber through the origin exactly at its isotropic conjugate points.*

From here and using Theorem 5.2, we have

Corollary 5.5. *Let γ be a geodesic in $M^3(\kappa,\tau)$ with slope angle $\theta \in]0,\pi[$. Then the following statements are equivalent:*

(i) *γ is isotropic;*
(ii) *γ does not admit any pair of conjugate points;*
(iii) *γ intersects each Hopf fiber only at a unique point.*

Remark 5.6. The Hopf fibers of $H_3(\tau)$ and $\widetilde{SL_2}(\kappa,\tau)$ are one-to-one geodesics and on $S^3(\kappa,\tau)$ they are simple closed geodesics with the same length (see [10]). For the Berger metric obtained upon scaling the fibers of the length 2π of the Hopf fibration $S^3(1) \to S^2(4) = S^3/S^1$, where $S^3(\kappa)$ or $S^2(\kappa)$ denote the correspoding spheres of radius $\frac{1}{\sqrt{\kappa}}$, the length of its fibers is $l = |2\pi c|$, c^2 being the ξ-sectional curvature. For $S^3(\kappa,\tau) \to S^2(\kappa)$, taking into account that metric of $S^2(\kappa)$ is given by a homothetic change from the metric of $S^2(4)$ with coefficient $\frac{4}{\kappa}$, it follows that l is given by $l = \frac{4\pi\tau}{\kappa}$.

Using the above results and taking into account that a geodesic without pairs of conjugate points is considered by definition strictly isotropic, we can conclude with the following corollaries.

Corollary 5.7. *On $H_3(\tau)$, we have:*

(i) *All geodesic is one-to-one.*
(ii) *A geodesic is isotropic if and only if its slope angle is $\theta = \pi/2$.*

Corollary 5.8. *On $\widetilde{SL_2}(\kappa,\tau)$, we have:*

(i) *All geodesic is one-to-one.*
(ii) *A geodesic is isotropic if and only if its slope angle θ belongs to $[\varepsilon, \pi-\varepsilon]$, where $\varepsilon = \arctan \frac{\tau}{\sqrt{-\kappa}}$.*

According with [7, Corollary 3.12], a geodesic on $S^3(\kappa,\tau)$ with slope angle $\theta \in]0,\pi[$ is closed if and only if $\frac{\tau^2-\kappa}{\sqrt{\lambda(\theta)}}\cos\theta \in \mathbb{Q}$. Then, using Proposition 5.3, it follows

Corollary 5.9. *On $S^3(\kappa,\tau)$, we have:*

(i) *A geodesic with slope angle $\theta \in]0,\pi[$ is closed if and only if its length l satisfies*

$$\frac{l}{2\pi}(\tau^2-\kappa)\cos\theta \in \mathbb{Q}.$$

(ii) *Any geodesic is not isotropic and it admits conjugate points.*

5.2. **Conjugate radius.** Denote by $\rho_{\mathrm{conj}}(\theta)$ the *conjugate radius* of a geodesic γ on $M^3(\kappa,\tau)$ with slope angle θ emanating from the origin. If $\rho_{\mathrm{conj}}(\theta)$ is finite, it is the distance along γ from the origin to its first conjugate point. Put $I_{M^3(\kappa,\tau)} = \{\theta \in [0,\pi] \mid \lambda(\theta) > 0\}$. Then, $I_{H_3(\tau)} = [0, \frac{\pi}{2}[\cup]\frac{\pi}{2}, \pi]$, $I_{S^3(\kappa,\tau)} = [0,\pi]$ and $I_{\widetilde{SL_2}(\kappa,\tau)} = [0,\varepsilon[\cup]\pi-\varepsilon,\pi]$, where $\varepsilon = \arctan\frac{\tau}{\sqrt{-\kappa}}$. Hence, one obtains $\mu(\theta) < 1/2$ for all $\theta \in I_{M^3(\kappa,\tau)}$ and moreover, $0 \le \mu(\theta) < 1/2$, if $\kappa < \tau^2$, and $\mu(\theta) \le 0$, if $\kappa > \tau^2$. See *Fig. 1* and *Fig. 2* for the behaviour of $f_\theta(s)$ in both cases.

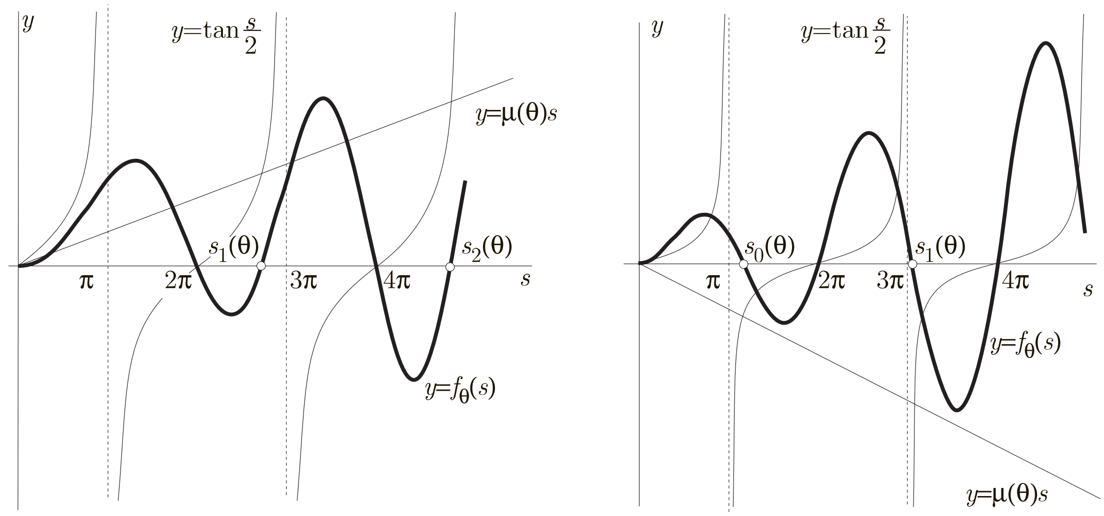

Fig.1 $\kappa < \tau^2$ $(0 \leqslant \mu(\theta) < 1/2)$ *Fig.2* $\kappa > \tau^2$ $(\mu(\theta) \leqslant 0)$

Proposition 5.10. *We have:*

(i) *The conjugate radius* $\rho_{\mathrm{conj}}(\theta)$ *satisfies*

$$\rho_{\mathrm{conj}}(\theta) = \begin{cases} +\infty & \text{if } \theta \notin I_{M^3(\kappa,\tau)}, \\ \frac{2\pi}{\sqrt{\lambda(\theta)}} & \text{if } \kappa < \tau^2,\ \theta \in I_{M^3(\kappa,\tau)}, \\ \frac{s_0(\theta)}{\sqrt{\lambda(\theta)}}, & \text{if } \kappa > \tau^2, \end{cases}$$

where $s_0(\theta) \in]\pi, 2\pi[$ *and* $\tan \frac{s_0(\theta)}{2} = \mu(\theta)s_0(\theta)$.

(ii) *The first conjugate point of* γ *is isotropic if and only if* $\kappa < \tau^2$ *and* $\theta \in]0, \pi[$.

Proof. Because $(\tan \frac{s}{2})'(0) = \frac{1}{2}$, the equation $\tan \frac{s}{2} = \mu(\theta)s$ has solutions in $]0, 2\pi[$ if and only if $\mu(\theta) \in]-\infty, 0[\cup]1/2, +\infty[$. Then, from Theorem 5.2, $\rho_{\mathrm{conj}}(\theta) = 2\pi/\sqrt{\lambda(\theta)}$ if $0 \leq \mu(\theta) < 1/2$ and $\rho_{\mathrm{conj}}(\theta) = s_0(\theta)/\sqrt{\lambda(\theta)}$ if $\mu(\theta) \in] - \infty, 0[$. It gives the result. $\square$

Remark 5.11. The first conjugate points to the origin for the Berger's spheres M^3_α given by $M^3_\alpha = S^3(4, 2\sin\alpha)$, $\alpha \in]0, \pi/2]$, has been determined by I. Chavel [5] (see also [18, Lemma 2.3], [17, Proposition 3.1]). This result is a particular case of the above proposition.

For $\kappa > \tau^2$, one obtains that $\theta = \pi/2$ minimises $\mu(\theta)$ and so it is also a minimum for the map s_0. Hence, the following result is immediate.

Corollary 5.12. *The conjugate radius* $\rho_{\mathrm{conj}}(M^3(\kappa,\tau))$ *of* $M^3(\kappa,\tau)$ *satisfies*

$$\rho_{conj}(M^3(\kappa,\tau)) = \begin{cases} \frac{2\pi}{\tau}, & \text{if } \kappa < \tau^2 \\ \frac{s_0(\frac{\pi}{2})}{\sqrt{\kappa}}, & \text{if } \kappa > \tau^2. \end{cases}$$

From Lemma 4.1, one directly obtains the following.

Proposition 5.13. *A simply connected non-symmetric* 3*-dimensional homogeneous manifold is normal if and only if all its first conjugate points of a fixed point are not isotropic.*

5.3. **Tangent conjugate locus.** Denote by s_p, $p \in \mathbb{N}$, the smooth map $s_p :]0,\pi[\to]2p\pi, 2(p+1)\pi[$ where $s_p(\theta) \in \mathcal{Z}^+(f_\theta)$, that is, $s_p(\theta)$ is the (unique) solution of the equation $\tan\frac{s}{2} = \mu(\theta)s$, such that $s_p(\theta) \in]2p\pi, 2(p+1)\pi[$. Then, as one can see in *Fig 1* and *Fig 2*, we get:

a) If $\kappa < \tau^2$, $\lim_{\theta\to 0^+} s_p(\theta) = \lim_{\theta\to\pi^-} s_p(\theta) = 2p\pi$, $s_p(\frac{\pi}{2})$ is a maximum for s_p on $]0,\pi[$ and $s_p(]0,\pi[) \subset]2p\pi, (2p+1)\pi[$.

b) If $\kappa > \tau^2$, $\lim_{\theta\to 0^+} s_p(\theta) = \lim_{\theta\to\pi^-} s_p(\theta) = 2(p+1)\pi$, $s_p(\frac{\pi}{2})$ is a minimum and $s_p(]0,\pi[) \subset](2p+1)\pi, 2(p+1)\pi[$.

We can extend s_p to $[0,\pi]$ taking $s_p(0) = s_p(\pi) = 2p\pi$, if $\kappa < \tau^2$, and $s_p(0) = s_p(\pi) = 2(p+1)\pi$, if $\kappa > \tau^2$. Moreover, if $\tau^2 - \kappa \to +\infty$, then $s_p(\theta) \to (2p+1)\pi^-$ and if $\tau^2 - \kappa \to -\infty$, then $s_p(\theta) \to (2p+1)\pi^-$, for all $\theta \in]0,\pi[$.

Denote by $\mathrm{conj}_{\mathrm{Isot}}(M^3(\kappa,\tau))$ the isotropic tangent conjugate locus of $M^3(\kappa,\tau)$. Then, using Theorem 5.2 and Proposition 5.10, we get

Lemma 5.14. *The tangent conjugate locus* $\mathrm{conj}(M^3(\kappa,\tau))$ *is the union of the following regular surfaces of revolution:*

$$\mathrm{conj}(M^3(\kappa,\tau)) = \begin{cases} \bigcup_{p\in\mathbb{N}} \mathcal{S}^1_p \cup \mathcal{S}^2_p, & \text{if } \kappa < \tau^2, \\ \mathcal{S}^2_0 \cup \Big(\bigcup_{p\in\mathbb{N}} \mathcal{S}^1_p \cup \mathcal{S}^2_p\Big), & \text{if } \kappa > \tau^2, \end{cases}$$

and

$$\mathrm{conj}_{\mathrm{Isot}}(M^3(\kappa,\tau)) = \bigcup_{p\in\mathbb{N}} \Big(\mathcal{S}^1_p \setminus \{(0,0,\pm\frac{2p\pi}{\tau})\}\Big),$$

where $\mathcal{S}^1_p = \{\frac{2p\pi}{\sqrt{\lambda(\theta)}}u(\theta,\phi)\}$, $\mathcal{S}^2_0 = \{\frac{s_0(\theta)}{\sqrt{\lambda(\theta)}}u(\theta,\phi)\}$ *and* $\mathcal{S}^2_p = \{\frac{s_p(\theta)}{\sqrt{\lambda(\theta)}}u(\theta,\phi)\}$, *for each* $p \in \mathbb{N}$, *and for all* $\theta \in I_{M^3(\kappa,\tau)}$ *and* $\phi \in [0,2\pi]$.

Remark 5.15. $\mathcal{S}^1_p$ is the surface generated by revolving the curve $\alpha^1_p(\theta) = (2p\pi/\sqrt{\lambda(\theta)})u(\theta)$, for $p \in \mathbb{N}$ and $\theta \in I_{M^3(\kappa,\tau)}$, $\mathcal{S}^2_p$ by revolving the curve $\alpha^2_p(\theta) = (s_p(\theta)/\sqrt{\lambda(\theta)})u(\theta)$ and $\mathcal{S}^2_0$, for $p = 0$. All these surfaces are regular. In fact, using that $\lim_{\theta\to 0^+} s_p'(\theta) = \lim_{\theta\to\pi^-} s_p'(\theta) = 0$ and putting $\mathfrak{m} = \mathbb{R}^3[x,y,z]$, where x, y, z are the cartesian coordinates with respect to $\{e_1, e_2, e_3\}$, we get

$$\begin{aligned} &\lim_{\theta\to 0^+} (\alpha^1_p)'(\theta) = -\lim_{\theta\to\pi^-} (\alpha^1_p)'(\theta) = (\tfrac{2p\pi}{\tau}, 0, 0) \\ &\lim_{\theta\to 0^+} (\alpha^2_p)'(\theta) = -\lim_{\theta\to\pi^-} (\alpha^2_p)'(\theta) = (\tfrac{2p\pi}{\tau}, 0, 0), \ \text{ if } \kappa < \tau^2, \\ &\lim_{\theta\to 0^+} (\alpha^2_p)'(\theta) = -\lim_{\theta\to\pi^-} (\alpha^2_p)'(\theta) = (\tfrac{2(p+1)\pi}{\tau}, 0, 0), \ \text{ if } \kappa > \tau^2. \end{aligned}$$

For the Heisenberg group $H_3(\tau)$, Lemma 5.14 gives

Proposition 5.16. *We have:*

(i) *The tangent conjugate locus* $\mathrm{conj}(H_3(\tau))$ *of* $H_3(\tau)$ *is given by the union the planes* $\Pi^\pm_p : z = \pm\frac{2\pi p}{\tau}$, $p \in \mathbb{N}$, *and the surfaces of revolution parameterised as*

$$\vec{x}^\pm_p(\theta,\phi) = \frac{s_p(\theta)}{\tau}(\tan\theta\cos\phi, \tan\theta\sin\phi, \pm 1), \quad \theta \in [0, \frac{\pi}{2}[.$$

(ii) $\mathrm{conj}_{\mathrm{Isot}}(H_3(\tau)) = \bigcup_{p\in\mathbb{N}} \Big(\Pi^\pm_p \setminus \{(0,0,\pm\frac{2p\pi}{\tau})\}\Big)$.

(iii) *The first tangent conjugate locus* $\mathrm{conj}^1(H^3(\tau))$ *of* $H_3(\tau)$ *are the planes* $\Pi_1^{\pm} : z = \pm\frac{2\pi}{\tau}$.
(iv) *The first conjugate points to the origin are all isotropic up to the points* $\gamma(\pm\frac{2\pi}{\tau})$, *where* γ *is the Hopf fiber through the origin.*

From Theorem 5.2, (x, y, z) belonging to $\mathrm{conj}(M^3(\kappa,\tau))$ satisfies

$$x = \frac{s \sin\theta\cos\phi}{\sqrt{\lambda(\theta)}}, \quad y = \frac{s \sin\theta\sin\phi}{\sqrt{\lambda(\theta)}}, \quad z = \frac{s\cos\theta}{\sqrt{\lambda(\theta)}},$$

for $\theta \in I_{M^3(\kappa,\tau)}$ and $s \in \mathcal{Z}^+(f_\theta)$. Then, we have $x^2+y^2+z^2 = \frac{s^2}{\lambda(\theta)}$ and for $\kappa \neq 0$, one obtains $\lambda(\theta) = \frac{\kappa s^2}{s^2-(\tau^2-\kappa)z^2}$. Hence, we get $\kappa(x^2+y^2)+\tau^2z^2 = s^2$. It implies that the surfaces $\mathcal{S}_p^1$, for $p \in \mathbb{N}$, in Lemma 5.14 are ellipsoids if $\kappa > 0$, and hyperboloids of two sheets if $\kappa < 0$, and $\mathcal{S}_p^2$ are surfaces of revolution generated by revolving the curve $\kappa x^2 + \tau^2 z^2 = s_p^2(\theta)$ about the z-axis for $p \in \mathbb{N}\cup\{0\}$. Therefore, one can conclude with the following results:

Proposition 5.17. *We have:*

(i) *The tangent conjugate locus* $\mathrm{conj}(S^3(\kappa,\tau))$ *of* $S^3(\kappa,\tau)$ *is given by the union of:*
  (a) *the ellipsoids* $\mathcal{E}_p : \ \kappa(x^2+y^2)+\tau^2z^2 = 4p^2\pi^2$, $p \in \mathbb{N}$;
  (b) *the surfaces of revolution* $\mathcal{S}_p$, *for all* $p \in \mathbb{N}$, *generated by revolving the curve* $\kappa x^2(\theta)+\tau^2z^2(\theta) = s_p^2(\theta)$ *about the z-axis and moreover, for* $p = 0$, *when* $\kappa > \tau^2$.

(ii) $\mathrm{conj}_{\mathrm{Isot}}(S^3(\kappa,\tau)) = \bigcup_{p\in\mathbb{N}} \left(\mathcal{E}_p \setminus \{(0,0,\pm\frac{2p\pi}{\tau})\}\right)$.
(iii) *If* $\kappa < \tau^2$, $\mathrm{conj}^1(S^3(\kappa,\tau)) = \mathcal{E}_1$.
(iv) *If* $\kappa > \tau$, $\mathrm{conj}^1(S^3(\kappa,\tau)) = \mathcal{S}_0$.

Proposition 5.18. *We have:*

(i) *The tangent conjugate locus* $\mathrm{conj}(\widetilde{SL}_2(\kappa,\tau))$ *of* $\widetilde{SL}_2(\kappa,\tau)$ *is given by the union of:*
  (a) *the hyperboloids of two sheets* $\mathcal{H}_p : \ \kappa(x^2+y^2)+\tau^2z^2 = 4p^2\pi^2$, $p \in \mathbb{N}$;
  (b) *the surfaces of revolution* $\mathcal{S}_p$, *for all* $p \in \mathbb{N}$, *generated by revolving the curve* $\kappa x^2(\theta)+\tau^2z^2(\theta) = s_p^2(\theta)$ *about the z-axis.*

(ii) $\mathrm{conj}_{\mathrm{Isot}}(\widetilde{SL}_2(\kappa,\tau)) = \bigcup_{p\in\mathbb{N}} \left(\mathcal{H}_p \setminus \{(0,0,\pm\frac{2p\pi}{\tau})\}\right)$.
(iii) $\mathrm{conj}^1(\widetilde{SL}_2(\kappa,\tau)) = \mathcal{H}_1$.
(iv) *The first conjugate points to the origin are all isotropic up to the points* $\gamma(\pm\frac{2\pi}{\tau})$, *where* γ *is the Hopf fiber through the origin.*

References

[1] T., Arias-Marco, A.M. and Naveira, The osculating rank of the Jacobi operator over g.o. spaces. A method for solving the Jacobi equation, preprint.
[2] J. Berndt and L. Vanhecke, Two natural generalizations of locally symmetric spaces, *Diff. Geom. Appl.* **2** (1992), 57-80.
[3] I. Chavel, Isotropic Jacobi fields, and Jacobi's equations on Riemannian homogeneous spaces, *Comment. Math. Helv.* **42** (1967), 237-248.
[4] I. Chavel, On normal Riemannian homogeneous spaces of rank 1, *Bull. Amer. Math. Soc.* **73** (1967), 477-481.
[5] I. Chavel, A class of Riemannian homogeneous spaces, *J. Diff. Geom.* **4** (1970), 13-20.
[6] B. Daniel, Isometric immersions into 3-dimensional homogeneous manifolds, to appear in *Comment. Math. Helv.* (http://arxiv.org/abs/math/0503500v1).

[7] S. Engel, On the geometry and trigonometry of homogeneous 3-manifolds with 4-dimensional isometry group, *Math. Z.* **254** (2006), 439-459.

[8] J.C. González-Dávila, Isotropic Jacobi fields on compact 3-symmetric spaces, to appear in *J. Diff. Geom.*

[9] J.C. González-Dávila, M.C. González-Dávila and L. Vanhecke, Normal flow space forms and their classification, *Publ. Math. Debrecen* **48** (1996), 151-173.

[10] J.C. González-Dávila, M.C. González-Dávila and L. Vanhecke, Classification of Killing-transversally symmetric spaces, *Tsukuba J. Math.* **20** (1996), 321-347.

[11] J.C. González-Dávila and A. M. Naveira, Jacobi fields and osculating rank of the Jacobi operator in some special classes of homogeneous Riemannian spaces, to appear in *Note di Matematica.*

[12] J.C. González-Dávila and R.O. Salazar, Isotropic Jacobi fields on naturally reductive spaces, *Publ. Math. Debrecen* **66** (2005), 41-61.

[13] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. II, Interscience Publishers, New York, 1969.

[14] O. Kowalski, Spaces with volume-preserving symmetries and related classes of Riemannian manifolds, *Rend. Sem. Mat. Univ. Politec. Torino*, Fascicolo Speciale Settembre 1983, 131-158.

[15] E. Macías, A.M. Naveira and A. Tarrío, The constant osculating rank of the Wilking manifold V_3, to appear in *C. R. Acad. Sci. Paris, Ser. I.*

[16] A.M. Naveira and A. Tarrío, A method for the resolution of the Jacobi equation $Y'' + RY = 0$ on the manifold $Sp(2)/SU(2)$, preprint.

[17] C. Rakotoniaina, Cut locus of the B-spheres, *Ann. Global Anal. Geom.* **3** (1985), 313-327.

[18] T. Sakai, Cut loci of Berger's spheres, *Hokkaido Math. J.* **10** (1981), 143-155.

[19] W.P. Thurston, *Three-dimensional geometry and topology*, Princeton Univ. Press, New Jersey, 1997.

[20] F. Tricerri and L. Vanhecke, *Homogeneous structures on Riemannian manifolds*, London Math. Soc. Lect. Note Series **83**, Cambridge Univ. Press, Cambridge, 1983.

[21] W. Ziller, The Jacobi equation on naturally reductive compact Riemannian homogeneous spaces, *Comment. Math. Helv.* **52** (1977), 573-590.

Department of Fundamental Mathematic, University of La Laguna, 38200 La Laguna, Tenerife, Spain

E-mail address: jcgonza@ull.es